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# Curvature Properties of Some Class of Minimal Hypersurfaces in Euclidean Spaces

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*Dedicated to the birthday of Professor Mileva Prvanović*

**Abstract.** We determine curvature properties of pseudosymmetry type of some class of minimal 2-quasi-umbilical hypersurfaces in Euclidean spaces $\mathbb{E}^{n+1}$, $n \geq 4$. We present examples of such hypersurfaces. The obtained results are used to determine curvature properties of biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$. Those hypersurfaces were recently investigated by Y. Fu in [38].

## 1. Introduction

Let $M$ be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geq 4$, where $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and $\widetilde{\kappa}$ are the sectional curvature and the scalar curvature of the ambient space, respectively. Let $\mathcal{U}_H \subset M$ be the set of all points at which the $(0,2)$-tensor $H^2$ is not expressed by a linear combination of the second fundamental tensor $H$ and the metric tensor $g$ of $M$. For precise definitions of the symbols used here, we refer to Section 2 of this paper (see also [19], [20] and [22]).

Curvature properties of pseudosymmetry type of hypersurfaces in semi-Riemannian spaces of constant curvature were investigated in several papers. In particular, hypersurfaces $M$ in $N_s^{n+1}(c)$, $n \geq 4$, with the tensor $H$ satisfying on $\mathcal{U}_H$

$$H^3 = \phi H^2 + \psi H + \rho g, \tag{1}$$

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for some functions $\phi$, $\psi$ and $\rho$, were investigated in the following papers: [9]–[13], [17]–[18], [21]–[23], [25], [28]–[31], [33], [36], [40], [48]–[52].

The main results of Section 3 are presented in Proposition 3.1 and Theorem 3.2. In Proposition 3.1 we present curvature properties of minimal hypersurfaces $M$ in $N^{n+1}_s(c)$, $n \geq 4$, satisfying (1). In Theorem 3.2 we present curvature properties of minimal hypersurfaces $M$ in semi-Euclidean spaces $\mathbb{E}^{n+1}_s$, $n \geq 4$, satisfying (1) with $\rho = 0$, i.e.

$$H^3 \;=\; \phi\, H^2 + \psi\, H. \tag{2}$$

We also present examples of hypersurfaces satisfying (1), see Example 3.1(iii) and Example 3.2(ii).

In Section 4 we consider hypersurfaces $M$ in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point of $\mathcal{U}_H \subset M$ exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ such that

$$\lambda_1 \;=\; 0,\;\; \lambda_2 \;=\; -(n-2)\lambda,\;\; \lambda_3 \;=\; \lambda_4 \;=\; \ldots \;=\; \lambda_n \;=\; \lambda \neq 0, \tag{3}$$

where $\lambda$ is a function on $\mathcal{U}_H$. Evidently, we have on $\mathcal{U}_H$: $tr(H) = 0$ and

$$H^3 \;=\; \phi\, H^2 + \psi\, H,\quad \phi \;=\; -(n-3)\,\lambda,\;\; \psi \;=\; (n-2)\,\lambda^2,\;\; \rho \;=\; 0. \tag{4}$$

In Proposition 4.1 we present curvature properties of hypersurfaces $M$ in $N^{n+1}(c)$, $n \geq 4$, satisfying (3). Using results of that proposition we obtain curvature properties of hypersurfaces $M$ in Euclidean spaces $\mathbb{E}^{n+1}$, $n \geq 4$, satisfying (3). We also present examples of hypersurfaces satisfying (3), see Example 4.1 and Example 4.2(ii). We recall that a Riemannian manifold $(M, g)$, $n = \dim M$, isometrically immersed in an $m$-dimensional Euclidean space $\mathbb{E}^m$ is said to be *biharmonic submanifold* ([6]) if its mean curvature vector field $\vec{H}$ satisfies $\Delta\vec{H} = 0$, where $\Delta$ is the Laplace operator of $M$. For recent survey on biharmonic submanifolds we refer to the book of B.-Y. Chen [6]. It is clear that any minimal submanifold in $\mathbb{E}^m$ is trivially biharmonic. A biharmonic submanifold in $\mathbb{E}^m$ is called *proper biharmonic* if it is not minimal. Very recently, biharmonic hypersurfaces with three distinct principal curvatures in $\mathbb{E}^5$ were investigated in [38]. In Theorem 3.2 of [38] it was stated that every biharmonic hypersurface $M$ with three distinct principal curvatures in $\mathbb{E}^5$ is minimal. The principal curvatures: $\lambda_1$, $\lambda_2$ and $\lambda_3$ of $M$ satisfy (3) with $n = 4$. In Theorem 4.3 we present curvature properties of those hypersurfaces.

## 2. Preliminaries

Throughout the paper all manifolds are assumed to be connected paracompact manifolds of class $C^\infty$. Let $(M, g)$ be an $n$-dimensional, $n \geq 3$, semi-Riemannian manifold and let $\nabla$ be its Levi-Civita connection and $\Xi(M)$ the Lie algebra of vector fields on $M$.

We define on $M$ the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of $\Xi(M)$, respectively, by

$$\begin{aligned}(X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y,\\ \mathcal{R}(X, Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\end{aligned}$$

where $A$ is a symmetric $(0, 2)$-tensor on $M$ and $X, Y, Z \in \Xi(M)$. The Ricci tensor $S$, the Ricci operator $\mathcal{S}$, the tensors $S^2$ and $S^3$ and the scalar curvature $\kappa$ of $(M, g)$ are defined by $S(X, Y) = tr\{Z \to \mathcal{R}(Z, X)Y\}$, $g(\mathcal{S}X, Y) = S(X, Y)$, $S^2(X, Y) = S(\mathcal{S}X, Y)$, $S^3(X, Y) = S^2(\mathcal{S}X, Y)$ and $\kappa = tr\,\mathcal{S}$, respectively. The endomorphisms $\mathcal{C}(X, Y)$ and $\mathrm{conh}(\mathcal{R})(X, Y)$ are defined by

$$\begin{aligned}\mathcal{C}(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z,\\ \mathrm{conh}(\mathcal{R})(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{n-2}(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y),\end{aligned}$$

respectively. Now the $(0,4)$-tensor $G$, the Riemann-Christoffel curvature tensor $R$, the Weyl conformal curvature tensor $C$ and the conharmonic tensor $\mathrm{conh}(R)$ of $(M,g)$ are defined by

$$\begin{aligned} G(X_1,X_2,X_3,X_4) &= g((X_1\wedge_g X_2)X_3,X_4),\\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4),\\ \mathrm{conh}(R)(X_1,X_2,X_3,X_4) &= g(\mathrm{conh}(\mathcal{R})(X_1,X_2)X_3,X_4), \end{aligned}$$

respectively, where $X_1,X_2,\ldots\in\Xi(M)$. We define the following subsets of $M$: $\mathcal{U}_R = \{x\in M\,|\,R-\frac{\kappa}{(n-1)n}G\neq 0 \text{ at } x\}$, $\mathcal{U}_S=\{x\in M\,|\,S-\frac{\kappa}{n}g\neq 0\text{ at }x\}$ and $\mathcal{U}_C=\{x\in M\,|\,C\neq 0\text{ at }x\}$. We note that $\mathcal{U}_S\cup\mathcal{U}_C=\mathcal{U}_R$.

Let $\mathcal{B}$ be a tensor field sending any $X,Y\in\Xi(M)$ to a skew-symmetric endomorphism $\mathcal{B}(X,Y)$, and let $B$ be a $(0,4)$-tensor associated with $\mathcal{B}$ by

$$B(X_1,X_2,X_3,X_4) = g(\mathcal{B}(X_1,X_2)X_3,X_4)\,. \tag{5}$$

The tensor $B$ is said to be a *generalized curvature tensor* if the following coditions are satisfied

$$\begin{aligned} &B(X_1,X_2,X_3,X_4) = B(X_3,X_4,X_1,X_2)\,,\\ &B(X_1,X_2,X_3,X_4)+B(X_3,X_1,X_2,X_4)+B(X_2,X_3,X_1,X_4) = 0\,. \end{aligned}$$

For $\mathcal{B}$ as above, let $B$ be again defined by (5). We extend the endomorphism $\mathcal{B}(X,Y)$ to a derivation $\mathcal{B}(X,Y)\cdot$ of the algebra of tensor fields on $M$, assuming that it commutes with contractions and $\mathcal{B}(X,Y)\cdot f = 0$, for any smooth function $f$ on $M$. For a $(0,k)$-tensor field $T$, $k\geq 1$, we can define the $(0,k+2)$-tensor $B\cdot T$ by

$$\begin{aligned} &(B\cdot T)(X_1,\ldots,X_k,X,Y) = (\mathcal{B}(X,Y)\cdot T)(X_1,\ldots,X_k)\\ &= -T(\mathcal{B}(X,Y)X_1,X_2,\ldots,X_k)-\cdots-T(X_1,\ldots,X_{k-1},\mathcal{B}(X,Y)X_k)\,. \end{aligned}$$

In addition, if $A$ is a symmetric $(0,2)$-tensor then we define the $(0,k+2)$-tensor $Q(A,T)$ by

$$\begin{aligned} &Q(A,T)(X_1,\ldots,X_k,X,Y) = (X\wedge_A Y\cdot T)(X_1,\ldots,X_k)\\ &= -T((X\wedge_A Y)X_1,X_2,\ldots,X_k)-\cdots-T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k)\,. \end{aligned}$$

The tensor $Q(A,T)$ is called the *Tachibana tensor of the tensors* $A$ and $T$, or shortly the Tachibana tensor (see, e.g., [23]). We mention that in some papers the tensor $Q(g,R)$ is called the Tachibana tensor ([41], [42], [43], [47]).

For a symmetric $(0,2)$-tensor $E$ and a $(0,k)$-tensor $T$, $k\geq 2$, we define their Kulkarni-Nomizu product $E\wedge T$ by ([18])

$$\begin{aligned} &(E\wedge T)(X_1,X_2,X_3,X_4;Y_3,\ldots,Y_k)\\ &= E(X_1,X_4)T(X_2,X_3,Y_3,\ldots,Y_k)+E(X_2,X_3)T(X_1,X_4,Y_3,\ldots,Y_k)\\ &\quad -E(X_1,X_3)T(X_2,X_4,Y_3,\ldots,Y_k)-E(X_2,X_4)T(X_1,X_3,Y_3,\ldots,Y_k). \end{aligned}$$

For instance, the following tensors are generalized curvature tensors: $R$, $C$, $G$, $conh(R)$ and $E\wedge F$, where $E$ and $F$ are symmetric $(0,2)$-tensors. For a symmetric $(0,2)$-tensor $E$ we define the $(0,4)$-tensor $\overline{E}$ by $\overline{E}=\frac{1}{2}E\wedge E$. In particular, we have $\overline{g}=G=\frac{1}{2}g\wedge g$ and

$$C = R-\frac{1}{n-2}g\wedge S+\frac{\kappa}{(n-2)(n-1)}G. \tag{6}$$

From (6) and the identity $Q(g,G)=0$ we get immediately

$$Q(g,C) = Q(g,R-\frac{1}{n-2}g\wedge S) = Q(g,conh(R)). \tag{7}$$

We also have

**Lemma 2.1.** *(cf. [27], Proposition 1) For any semi-Riemannian manifold $(M, g)$, $n \geq 4$, the following identities hold good*

$$\begin{aligned} conh(R) \cdot S &= C \cdot S - \frac{\kappa}{(n-2)(n-1)}\, Q(g, S), \\ R \cdot conh(R) &= R \cdot C, \\ conh(R) \cdot R &= C \cdot R - \frac{\kappa}{(n-2)(n-1)}\, Q(g, R), \\ conh(R) \cdot conh(R) &= C \cdot C - \frac{\kappa}{(n-2)(n-1)}\, Q(g, C). \end{aligned} \tag{8}$$

For a symmetric $(0, 2)$-tensor $A$ we define the endomorphism $\mathcal{A}$ and the tensors $A^2$ and $A^3$ by $g(\mathcal{A}X, Y) = A(X, Y)$, $A^2(X, Y) = A(\mathcal{A}X, Y)$ and $A^3(X, Y) = A^2(\mathcal{A}X, Y)$, respectively.

**Lemma 2.2.** *Let $E_1$, $E_2$ and $F$ be symmetric $(0, 2)$-tensors at a point $x$ of a semi-Riemannian manifold $(M, g)$, $n \geq 3$.*
*(i) ([17], [18]) At $x$ we have*

$$E_1 \wedge Q(E_2, F) + E_2 \wedge Q(E_1, F) + Q(F, E_1 \wedge E_2) = 0\, .$$

*In particular, if $E = E_1 = E_2$ then at $x$ we have*

$$E \wedge Q(E, F) = -Q(F, \overline{E})\, .$$

*Moreover (see, e.g., [21], Section 3)*

$$Q(E, E \wedge F) = -Q(F, \overline{E})\, .$$

*(ii) ([44], Lemma 3.2) At $x$ we have*

$$\begin{aligned} &G \cdot F = Q(g, F), \;\; (g \wedge F) \cdot F = Q(g, F^2), \\ &-(g \wedge F) \cdot (g \wedge F) = Q(F^2, G)\, . \end{aligned}$$

*Moreover, if $A$ is a symmetric $(0, 2)$-tensor and $B$ a generalized curvature tensor then*

$$G \cdot A = Q(g, A)\, , \qquad G \cdot B = Q(g, B)\, .$$

*(iii) (see, e.g., [37], Lemma 2.4 (iii)) At $x$ we have*

$$Q(E_1, E_2 \wedge F) + Q(E_2, F \wedge E_1) + Q(F, E_1 \wedge E_2) = 0\, .$$

As an immediate consequence of (6) and Lemma 2.2(ii) we get (also see [28], p. 217)

**Lemma 2.3.** *On any semi-Riemannian manifold $(M, g)$, $n \geq 4$, we have the following identity*

$$C \cdot S = R \cdot S - \frac{1}{n-2}\, Q(g, S^2 - \frac{\kappa}{n-1}\, S)\, . \tag{9}$$

Let $B_{hijk}$, $T_{hijk}$, and $A_{ij}$ be the local components of generalized curvature tensors $B$ and $T$ and a symmetric $(0, 2)$-tensor $A$ on $M$, respectively, where $h, i, j, k, l, m, p, q \in \{1, 2, \ldots, n\}$. The local components $(B \cdot T)_{hijklm}$ and $Q(A, T)_{hijklm}$ of the tensors $B \cdot T$, $Q(A, T)$, $B \cdot A$ and $Q(g, A)$ are the following

$$\begin{aligned} (B \cdot T)_{hijklm} &= g^{pq}(T_{pijk}B_{qhlm} + T_{hpjk}B_{qilm} + T_{hipk}B_{qjlm} + T_{hijp}B_{qklm}), \\ Q(A, T)_{hijklm} &= A_{hl}T_{mijk} + A_{il}T_{hmjk} + A_{jl}T_{himk} + A_{kl}T_{hijm} \\ &\quad - A_{hm}T_{lijk} - A_{im}T_{hljk} - A_{jm}T_{hilk} - A_{km}T_{hijl}, \\ (B \cdot A)_{hklm} &= g^{pq}(A_{pk}B_{qhlm} + A_{ph}B_{qklm}), \\ Q(g, A)_{hklm} &= g_{hl}A_{km} + g_{kl}A_{hm} - g_{hm}A_{kl} - g_{km}A_{hl}. \end{aligned}$$

The manifold $(M, g)$, $n \geq 3$, is said to be an *Einstein manifold* [1] if $S = \frac{\kappa}{n}\, g$ on $M$.

Einstein manifolds form a subclass of the class of quasi-Einstein manifolds. The semi-Riemannian manifold $(M, g)$, $n \geq 3$, is called a *quasi-Einstein manifold* if rank $(S - \alpha\, g) = 1$ on $\mathcal{U}_S$, where $\alpha$ is some function on this set. Every warped product manifold $\overline{M} \times_F \widetilde{N}$ of an 1-dimensional $(\overline{M}, \overline{g})$ base manifold and an 2-dimensional manifold $(\tilde{N}, \tilde{g})$ or an $(n-1)$-dimensional Einstein manifold $(\tilde{N}, \tilde{g})$, $n \geq 4$, with a warping function $F$, is a quasi-Einstein manifold. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations and investigation of quasi-umbilical hypersurfaces of conformally flat spaces. Quasi-Einstein hypersurfaces in semi-Riemannian spaces of constant curvature were studied among others in: [17], [21], [25], [31] and [40], see also [20]. We refer to [8] and [27] for recent results on quasi-Einstein manifolds.

The semi-Riemannian manifold $(M, g)$, $n \geq 3$, is called a *2-quasi-Einstein manifold* if rank $(S - \alpha\, g) \leq 2$ on $U_S$ and rank $(S - \alpha\, g) = 2$ on some open non–empty subset of $U_S$, where $\alpha$ is some function on $U_S$. It is clear that every warped product manifold $\overline{M} \times_F \widetilde{N}$ of an 2-dimensional $(\overline{M}, \overline{g})$ base manifold and an 2-dimensional manifold $(\tilde{N}, \tilde{g})$ or an $(n2)$dimensional Einstein manifold $(\tilde{N}, \tilde{g})$, $n \geq 5$, with a warping function $F$, is a 2-quasi-Einstein manifold. Therefore some exact solutions of the Einstein field equations are 2-quasi-Einstein manifolds, e.g. the Reissner-Nordström-de Sitter type spacetimes are such manifolds (see, e.g., [44]). It seems that the Reissner-Nordström spacetime is the "oldest" example of a 2-quasi-Einstein warped product manifold. It is easy to see that every 2-quasi-umbilical hypersurface in a space of constant curvature is a 2-quasi-Einstein manifold (see Remark 3.1). We refer to [24] for recent results on 2-quasi-Einstein warped product manifolds.

## 3. Hypersurfaces in spaces of constant curvature

Let $M$ be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, \widetilde{g})$ of dimension $n+1$, $n \geq 3$. Let $g$ be the metric tensor induced on $M$ from $\widetilde{g}$. Let $\nabla$ and $\widetilde{\nabla}$ be the Levi-Civita connections corresponding to the metric tensors $g$ and $\widetilde{g}$, respectively. We denote by $\xi$ a local unit normal vector field on $M$ in $N$ and let $\varepsilon = \widetilde{g}(\xi, \xi) = \pm 1$. We can write the *Gauss formula* and the *Weingarten formula* of $(M, g)$ in $(N, \widetilde{g})$ in the form: $\widetilde{\nabla}_X Y = \nabla_X Y + \varepsilon\, H(X, Y)\, \xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where $X, Y$ are vector fields tangent to $M$. $H$ is the *second fundamental tensor* and $\mathcal{A}$ the *shape operator* of $(M, g)$ in $(N, \widetilde{g})$. We have $H(X, Y) = g(\mathcal{A}X, Y)$, for any vectors fields $X, Y$ tangent to $M$. Further, we set $H^p(X, Y) = g(\mathcal{A}^p X, Y)$, $p = 1, 2, \ldots$, $H^1 = H$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by $H^p_{hk}$ the local components of the tensor $H^p$.

According to [4], [5], [7], [46], [53], a hypersurface $M$ in an $(n+1)$-dimensional Riemannian manifold $N$ is said to be *quasi-umbilical*, resp., *2-quasi-umbilical*, at a point $x \in M$ if it has a principal curvature with multiplicity $n-1$, resp., $n-2$, i.e. when the principal curvatures of $M$ at $x$ are given by $\lambda_1$, $\lambda_2 = \lambda_3 = \ldots = \lambda_n$, resp., $\lambda_1$, $\lambda_2$, $\lambda_3 = \lambda_4 = \ldots = \lambda_n$. If $M$ is a hypersurface in an $(n+1)$-dimensional semi-Riemannian manifold $N$ then $M$ is called *quasi-umbilical* (see, e.g., [34], [40]), resp., *2-quasi-umbilical* (see, e.g., [36], [40]), at a point $x \in M$ when rank $(H - \alpha\, g) = 1$, resp., rank $(H - \alpha\, g) = 2$, holds at $x$, for some $\alpha \in \mathbb{R}$.

We recall that a hypersurface $M$ in a semi-Riemannian conformally flat manifold $N$ is quasi-umbilical at a point $x \in M$ if and only if its Weyl conformal curvature tensor $C$ vanishes at this point ([34], Theorem 4.1). Thus a point $x \in M$ is a non-quasi-umbilical point of $M$ if and only if the tensor $C$ is non-zero at $x$, i.e. $x \in \mathcal{U}_C \subset M$.

We denote by $R$ and $\widetilde{R}$ the Riemann-Christoffel curvature tensors of $(M, g)$ and $(N, \widetilde{g})$, respectively. Let $x^r = x^r(y^k)$ be the local parametric expression of $(M, g)$ in $(N, \widetilde{g})$, where $y^k$ and $x^r$ are local coordinates of $M$ and $N$, respectively, $h, i, j, k \in \{1, 2, \ldots, n\}$ and $p, r, t, u \in \{1, 2, \ldots, n+1\}$. The *Gauss equation* of $(M, g)$ in $(N, \widetilde{g})$ reads

$$R_{hijk} \;=\; \widetilde{R}_{prtu} B_h^{\,p} B_i^{\,r} B_j^{\,t} B_k^{\,u} + \varepsilon\, (H_{hk} H_{ij} - H_{hj} H_{ik}), \quad B_k^{\,r} \;=\; \frac{\partial x^r}{\partial y^k}, \tag{10}$$

where $\widetilde{R}_{prtu}$, $R_{hijk}$ and $H_{hk}$ are the local components of the tensors $\widetilde{R}$, $R$ and $H$, respectively.

Let $M$ be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$ with signature $(s, n+1-s)$, $n \geq 4$, where $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and $\widetilde{\kappa}$ are the sectional curvature and the scalar curvature of the ambient space, respectively. Now (10) turns into

$$R_{hijk} \;=\; \varepsilon\,(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\widetilde{\kappa}}{n(n+1)}\,G_{hijk},\;\; \varepsilon \;=\; \pm 1. \tag{11}$$

Contracting (11) with $g^{ij}$ and $g^{kh}$ we obtain

$$S_{hk} \;=\; \varepsilon\,(tr(H)\,H_{hk} - H^2_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g_{hk}, \tag{12}$$

$$\kappa \;=\; \varepsilon\,((tr(H))^2 - tr(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1}, \tag{13}$$

respectively, where $tr(H^2) = g^{hk}H^2_{hk}$ and $S_{hk}$ are the local components of the Ricci tensor $S$ of $M$. It is known that on $M$ we have ([34])

$$R\cdot R - Q(S,R) \;=\; -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}\,Q(g,C). \tag{14}$$

In particular, if the ambient space is a semi-Euclidean space $\mathbb{E}^{n+1}_s$ then (14) reduces to

$$R\cdot R \;=\; Q(S,R). \tag{15}$$

Let $M$ be a hypersurface in $N^{n+1}_s(c)$, $n\geq 4$, satisfying (1) on $\mathcal{U}_H$. We define on $\mathcal{U}_H$ the following functions ([48], eq. (34)):

$$\begin{aligned}
\beta_1 &= \varepsilon\,(\phi - tr(H)),\\
\beta_2 &= -\frac{\varepsilon}{n-2}(\phi\,(2tr(H)-\phi) - (tr(H))^2 - \psi - (n-2)\varepsilon\mu),\\
\beta_3 &= \varepsilon\mu tr(H) + \frac{1}{n-2}\,(\psi\,(2tr(H)-\phi) + (n-3)\rho),\\
\beta_4 &= \beta_3 - \varepsilon\beta_2 tr(H) + \frac{(n-1)\widetilde{\kappa}\beta_1}{n(n+1)},
\end{aligned} \tag{16}$$

where the functions: $\phi$, $\psi$, $\rho$ $\mu$ are defined by (1) and

$$\mu \;=\; \frac{1}{n-2}\left(\frac{\kappa}{n-1} - \frac{\widetilde{\kappa}}{n+1}\right), \tag{17}$$

respectively. We also have on $\mathcal{U}_H$ ([48], eqs. (43), (52), (45), (46)):

$$R\cdot S \;=\; \frac{\widetilde{\kappa}}{n(n+1)}\,Q(g,S) + \rho\,Q(g,H) - \varepsilon\beta_1\,Q(H,H^2), \tag{18}$$

$$C\cdot S \;=\; \beta_1\,Q(H,S) + \beta_2\,Q(g,S) + \beta_4\,Q(g,H), \tag{19}$$

$$\begin{aligned}
(n-2)\,R\cdot C \;=\;& (n-2)\,Q(S,R)\\
&-\frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}\,Q(g,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G)\\
&+\rho\,Q(H,G) + (\phi - tr(H))\,g\wedge Q(H,H^2),
\end{aligned} \tag{20}$$

$$\begin{aligned}
(n-2)\,C\cdot R \;=\;& (n-3)\,Q(S,R)\\
&+\left(\frac{\kappa}{n-1} + \varepsilon\psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right)Q(g,R)\\
&-\frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G) + (\phi - tr(H))\,H\wedge Q(g,H^2),
\end{aligned} \tag{21}$$

where $\beta_1,\ldots,\beta_4$ are defined by (16).

**Example 3.1.** (i) (Example 1.1, [54]) The Clifford hypersurfaces in $N^n(c)$, $c \neq 0$, $n \geq 4$. (a) For $c > 0$ we set $N^n(c) = S^n(c) = \{x \in \mathbb{R}^{n+1} :< x, x >= \frac{1}{c}\}$, where $< \cdot, \cdot >$ is the standard inner product on $\mathbb{R}^{n+1}$. For $1 \leq m \leq n-2$, $t \in (0, \frac{\pi}{2})$, let $M_{m,n-m-1}(c,t) = S^m(\frac{c}{\sin^2 t}) \times S^{n-m-1}(\frac{c}{\cos^2 t})$. We view $x = (x_1, x_2) \in M_{m,n-m-1}(c,t)$ as a vector in $\mathbb{R}^{n+1} = \mathbb{R}^{m+1} \times \mathbb{R}^{n-m}$, then $x \in S^n(c)$. This is the standard isometric embedding of $M_{m,n-m-1}(c,t)$ into $S^n(c)$. In this situation, for suitably chosen unit normal vector field, $M_{m,n-m-1}(c,t)$ has two distinct principal curvatures $\rho_1 = \sqrt{c} \cot t$ of the multiplicity $m$ and $\rho_2 = -\sqrt{c} \tan t$ of the multiplicity $n-m-1$.
(b) For $c < 0$ we set $N^n(c) = H^n(c) = \{x \in \mathbb{R}^{n+1}_1 :< x, x >_1 = \frac{1}{c}, x^{n+1} > 0\}$. Here $< x, y >_1 = x^1 y^1 + \cdots + x^n y^n - x^{n+1} y^{n+1}$ is the standard Lorentzian inner product on $\mathbb{R}^{n+1}_1$. For $1 \leq m \leq n-2$, $t \in (0, +\infty)$, let $M_{m,n-m-1}(c,t) = S^m(\frac{-c}{\sinh^2 t}) \times H^{n-m-1}(\frac{c}{\cosh^2 t})$. Then $M_{m,n-m-1}(c,t)$ is an embedded hypersurface in $H^n(c)$, and for suitably chosen unit normal vector field, it has two distinct principal curvatures $\rho_1 = -c \coth t$ of the multiplicity $m$ and $\rho_2 = -c \tanh t$ of the multiplicity $n-m-1$.
(ii) (a) If $2 \leq m \leq n-3$ and $(m-1)\, c_1 \neq (n-m-2)\, c_2$, where $c_1 = \frac{c}{\sin^2 t}$, $c_2 = \frac{c}{\cos^2 t}$, $t \in (0, \frac{\pi}{2})$ then in view of Proposition 3.4 of [39] the Riemann-Christoffel curvature tensor $R$ of $M_{m,n-m-1}(c,t)$ is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$, i.e. $M_{m,nm1}(c,t)$ is a Roter type hypersurface.
(b) If $2 \leq m \leq n-3$ and $(m-1)\, c_1 \neq (n-m-2)\, c_2$, where $c_1 = \frac{-c}{\sinh^2 t}$, $c_2 = \frac{c}{\cosh^2 t}$, $t \in (0, +\infty)$, then in view of Proposition 3.4 of [39] the Riemann-Christoffel curvature tensor $R$ of $M_{m,n-m-1}(c,t)$ is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and and $S \wedge S$, i.e. $M_{m,n-m-1}(c,t)$ is a Roter type hypersurface. (c) The Roter type manifolds (and in particular, hypersurfaces in space forms) were studied among others in the papers: [19], [20], [22], [25], [26], [32], [39] and [44].
(iii) Let $M$ be a $n$-dimensional hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$. Precisely, let $M$ the cone over the Clifford hypersurface $M_{m,n-m-1}(c,t)$ defined in (i). We refer to Section 3 of [45] for precise definition and properties of cones. In particular, from Section 3 of [45] it follows immediately that $M$ has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{t}\rho_1$ and $\lambda_3 = \frac{1}{t}\rho_2$, $t \in \mathbb{R}^+$, of the multiplicity 1, $m$ and $n-m-1$, respectively. Thus we see that the cone over the Clifford hypersurface $M_{m,n-m-1}(c,t)$, presented in (i) is a hypersurface in $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures and satisfying at every point $\mathcal{U}_H = M$ the equation (1) with $\rho = 0$, i.e. (2).
(iv) We mention that an example of a hypersurface $M$ in $\mathbb{E}^{n+1}$, $n \geq 4$, satisfying (1) on $\mathcal{U}_H \subset M$, with non-zero function $\rho$ and $\phi = tr(H)$, is presented in [52].
(v) The Cartan hypersurfaces of dimension 6, 12 or 24 satisfy (2), with $\phi = tr(H) = 0$. Curvature properties of these hypersurfaces are presented in [18] (Theorem 4.3).

**Proposition 3.1.** *If $M$ is a minimal hypersurface in a semi-Riemannian space of constant curvature $N^{n+1}_s(c)$, $n \geq 4$,* *satisfying (1) on $\mathcal{U}_H \subset M$ then the folowing conditions are satisfied on this set: (14) and*

$$\begin{aligned}
\beta_1 &= \varepsilon\,\phi, \quad \beta_2 = \frac{\varepsilon}{n-2}(\phi^2+\psi+(n-2)\varepsilon\mu),\\
\beta_3 &= \frac{1}{n-2}\,((n-3)\rho-\psi\,\phi), \quad \beta_4 = \beta_3+\frac{(n-1)\widetilde{\kappa}\varepsilon\,\phi}{n(n+1)}, \qquad (22)\\
R\cdot S &= \frac{\widetilde{\kappa}}{n(n+1)}\,Q(g,S)+\rho\,Q(g,H)-\phi\,Q(H,H^2), \qquad (23)\\
C\cdot S &= \varepsilon\,\phi\,Q(H,S)+\beta_2\,Q(g,S)+\beta_4\,Q(g,H), \qquad (24)\\
(n-2)\,R\cdot C &= (n-2)\,Q(S,R)\\
&\quad -\frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}\,Q(g,R)-\frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G)\\
&\quad +\rho\,Q(H,G)+\phi\,g\wedge Q(H,H^2), \qquad (25)\\
(n-2)\,C\cdot R &= (n-3)\,Q(S,R)\\
&\quad +\left(\frac{\kappa}{n-1}+\varepsilon\psi-\frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)}\right)Q(g,R)\\
&\quad -\frac{(n-3)\widetilde{\kappa}}{n(n+1)}\,Q(S,G)+\phi\,H\wedge Q(g,H^2), \qquad (26)\\
(\phi\psi+\rho)\,H &= A^2+\varepsilon(\phi^2+\psi)\,A-\phi\rho\,g, \qquad (27)\\
A^3 &= -\varepsilon(\phi^2+2\psi)\,A^2+(2\phi\rho-\psi^2)\,A-\varepsilon\rho^2\,g, \qquad (28)\\
(\phi\psi+\rho)^2\,R &= \frac{\varepsilon}{2}\,(A^2+\varepsilon(\phi^2+\psi)\,A-\phi\rho\,g)\wedge(A^2+\varepsilon(\phi^2+\psi)\,A-\phi\rho\,g)\\
&\quad +\frac{(\phi\psi+\rho)^2\widetilde{\kappa}}{n(n+1)}\,G, \qquad (29)
\end{aligned}$$

*where $\beta_1,\ldots,\beta_4$ are defined by (22) and*

$$A = S-\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\,g. \qquad (30)$$

**Proof.** Since $M$ is a minimal hypersurface, (16) and (18)-(21) turn into (22)-(26), respectively. From (1), (12) and (30) we get easily

$$\begin{aligned}
A &= -\varepsilon\,H^2, \quad A^2 = H^4, \quad A^3 = -\varepsilon\,H^6, \qquad (31)\\
H^4 &= (\phi^2+\psi)\,H^2+(\phi\psi+\rho)\,H+\phi\rho\,g, \qquad (32)\\
H^6 &= (\phi^2+\psi)\,H^4+\phi(\phi\psi+2\rho)\,H^2+\psi(\phi\psi+\rho)\,H+\rho(\phi\psi+\rho)\,g. \qquad (33)
\end{aligned}$$

Now (27)-(29) are immediate consequences of (11) and (31)-(33). Our proposition is thus proved.

**Remark 3.1.** (i) Let $M$ be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $\mathcal{U}_H \subset M$ we have exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$, then (18)-(21) hold on $\mathcal{U}_H$ with $\varepsilon = 1$ and

$$\phi = \lambda_1+\lambda_2+\lambda_3, \quad \psi = -\lambda_1\lambda_2-\lambda_1\lambda_3-\lambda_2\lambda_3, \quad \rho = \lambda_1\lambda_2\lambda_3. \qquad (34)$$

(ii) Let $M$ be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $\mathcal{U}_H \subset M$ we have exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda$, then from (12) it follows that

$$\mathrm{rank}\left(S-\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}+\lambda\,(tr(H)-\lambda)\right)g\right) = 2 \qquad (35)$$

on $\mathcal{U}_H$. Moreover, the following condition holds on $\mathcal{U}_H$ (see [36], p. 53)

$$C \cdot C = -\frac{(n-3)\lambda_1\lambda_2}{(n-1)(n-2)} Q(g,C). \tag{36}$$

We refer to [13], [35], [19], [27] and [32] for results on semi-Riemannian manifolds $(M, g)$, $\dim M \geq 4$, and in particular, on hypersurfaces $M$ in $N_s^{n+1}(c)$, $n \geq 4$, satisfying on $\mathcal{U}_C \subset M$

$$C \cdot C = L\, Q(g,C), \tag{37}$$

where $L$ is some function on this set. We mention that the warped product manifold $\overline{M} \times_F \widetilde{N}$, of manifolds $(\overline{M}, \overline{g})$, $\dim \overline{M} = 2$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = 2$, and the warping function $F$ satisfies (37) on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$ (see, e.g., [20]). We also mention that the warped product manifold $\overline{M} \times_F \widetilde{N}$, of manifolds $(\overline{M}, \overline{g})$, $\dim \overline{M} = 1$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = 3$, and the warping function $F$ satisfies on $\mathcal{U}_C \subset \overline{M} \times_F \widetilde{N}$

$$R \cdot R - Q(S,R) = L\, Q(g,C),$$

where $L$ is some function on this set ([11]).

Proposition 3.1 leads to the following

**Theorem 3.2.** *If $M$ is a minimal hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying (2) on $\mathcal{U}_H \subset M$ then the following conditions are satisfied on this set: (15) and*

$$\begin{aligned}
\phi\psi H &= S^2 + \varepsilon(\phi^2 + \psi)\, S,\\
S^3 &= -\varepsilon(\phi^2 + 2\psi)\, S^2 - \psi^2\, S,\\
(\phi\psi)^2 R &= \frac{\varepsilon}{2}(S^2 + \varepsilon(\phi^2+\psi)\, S) \wedge (S^2 + \varepsilon(\phi^2+\psi)\, S),\\
R \cdot S &= \varepsilon\phi\, Q(H,S),\\
C \cdot S &= \varepsilon\,\phi\, Q(H,S) - \frac{\psi\phi}{n-2} Q(g,H) + \frac{\varepsilon}{n-2}(\phi^2 + \psi + \frac{\varepsilon\kappa}{n-1})\, Q(g,S),\\
(n-2)\, R \cdot C &= (n-2)\, Q(S,R) - \varepsilon\phi\, g \wedge Q(H,S),\\
(n-2)\, C \cdot R &= (n-3)\, Q(S,R) + (\varepsilon\psi + \frac{\kappa}{n-1})\, Q(g,R) - \varepsilon\phi\, H \wedge Q(g,S).
\end{aligned}$$

**Example 3.2.** (i) Let $\mathcal{M}$ be a $(n-1)$-dimensional hypersurface in $n$-dimensional standard unit sphere $S^n(1)$ in the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$. Precisely, let $\mathcal{M}$ be the Clifford torus $S^p(c_1) \times S^{n-p-1}(c_2)$, $c_1 = r_1^{-1}$, $c_2 = r_2^{-1}$, $r_1 = \sqrt{\frac{p}{n-1}}$, $r_2 = \sqrt{\frac{n-p-1}{n-1}}$, $1 \leq p \leq n-2$. It is well-known that $\mathcal{M}$ is a minimal hypersurface of $S^n(1)$ having at every point exactly two principal curvatures $\rho_1$ and $\rho_2$ of the multiplicity $p$ and $n-p-1$, respectively, satisfying

$$\rho_1\rho_2 + 1 = 0, \quad \rho_i^2 = r_i^{-2} - 1, \quad i = 1, 2. \tag{38}$$

(ii) Let $M$ be a $n$-dimensional hypersurface in the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$. Precisely, let $M$ be the cone over $\mathcal{M}$. We refer to Section 3 of [45] for precise definition and properties of such hypersurfaces. In particular, from Section 3 of [45] it follows immediately that $M$ has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{t}\rho_1$ and $\lambda_3 = \frac{1}{t}\rho_2$, $t \in \mathbb{R}^+$, of the multiplicity 1, $p$ and $n-p-1$, respectively. Thus we see that the cone $M$ over the Clifford torus $S^p(c_1) \times S^{n-p-1}(c_2)$ is a hypersurface in $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures satisfying at every point (2). Using (38) we can check that $\psi = -\lambda_2\lambda_3 = t^{-2}$ and

$$\begin{aligned}
\phi^2 &= (\lambda_2 + \lambda_3)^2 = \frac{1}{t^2}(\rho_1 + \rho_2)^2 = \frac{1}{t^2}(\rho_1^2 + \rho_2^2 - 2) = \frac{1}{t^2}\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} - 4\right)\\
&= \frac{1}{t^2}\left(\frac{(n-1)^2}{p(n-p-1)} - 4\right) = \frac{((n-p-1)+p)^2 - 4p(n-p-1)}{p(n-p-1)t^2} = \frac{(n-2p-1)^2}{p(n-p-1)t^2}.
\end{aligned}$$

If $p \neq n-p-1$ then in view of Theorem 3.2 the Riemann-Christoffel curvature tensor $R$ of the cone $M$ is expressed at every point by a linear combination of the tensors $g \wedge g$, $g \wedge S$ and $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. We refer to [50] and [52] for further results on hypersurfaces with the curvature tensor having the above presented property.

**Remark 3.2.** (i) Let $M$ be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and let the condition

$$H^3 = tr(H)\, H^2 + \psi\, H + \rho\, g,$$

be satisfied on $\mathcal{U}_H \subset M$, where $\psi$ and $\rho$ are some functions on this set. Using the identity (9), and (3.6) and (3.7) of [23] we get on $\mathcal{U}_H$

$$C \cdot S = \left( \varepsilon\psi + \frac{\kappa}{(n-2)(n-1)} - \frac{(2n-3)\widetilde{\kappa}}{n(n+1)} \right) Q(g,S) + \frac{n-3}{n-2}\, Q(g,S^2). \tag{39}$$

(ii) (cf., [29], Lemma 4.2) Let $M$ is a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying on $\mathcal{U}_H \subset M$ the relation

$$H^3 = tr(H)\, H^2 - \frac{\varepsilon\kappa}{n-1}\, H.$$

Now (39), by (3.9) of [23], and the conditions $\widetilde{\kappa} = 0$ and $\psi = -\frac{\varepsilon\kappa}{n-1}$, reduces on $\mathcal{U}_H$ to

$$C \cdot S = 0. \tag{40}$$

Hypersurfaces satisfying (40) were investigated among others in: [2], [9]–[12], [21], [28]–[30].
(iii) Let $(M,g)$, $n \geq 4$, be a non-Riemannian semi-Riemannian manifolds with parallel Weyl tensor ($\nabla C = 0$), which are in addition non-locally symmetric ($\nabla R \neq 0$) and non-conformally flat ($C \neq 0$). Such manifolds are called essentially conformally symmetric manifolds, e.c.s. manifolds, in short (see, e.g., [14]). Certain e.c.s. metrics are realized on compact manifolds ([15], [16]). As it was stated in [14], e.c.s. manifolds are semisymmetric manifolds ($R \cdot R = 0$) satisfying: $\kappa = 0$ , $S^2 = 0$ and $C(\widetilde{S}X_1, X_2, X_3, X_4) = 0$, for any $X_1, \ldots, X_4 \in \Xi(M)$. Thus, in view of Lemma 2.3, we see that (40) holds on every e.c.s. manifold.

## 4. Some special minimal 2-quasi-umbilical hypersurfaces

In this section we consider hypersurfaces $M$ in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having at every point of $\mathcal{U}_H \subset M$ exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3 = \lambda$ such that (3) is satisfied. Thus at every point of $\mathcal{U}_H$ we have: $\lambda \neq 0$, $tr(H) = 0$ and

$$\operatorname{rank}(H - \lambda\, g) = 2, \tag{41}$$

$$\operatorname{rank}\left( S - \left( \frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \lambda^2 \right) g \right) = 2. \tag{42}$$

The last condition follows immediately from (35). Therefore $\mathcal{U}_H$ is a minimal, 2-quasi-umbilical and 2-quasi Einstein open submanifold of $M$. Evidently, (36) reduces to

$$C \cdot C = 0. \tag{43}$$

This, together with (7) and (8), yields

$$conh(R) \cdot conh(R) = -\frac{\kappa}{(n-2)(n-1)}\, Q(g, conh(R)). \tag{44}$$

Furthermore (1), (12), (13), (16) and (18)-(34) give (4) and

$$S = -H^2 + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g\,, \quad \kappa = -tr(H^2) + \frac{(n-1)\widetilde{\kappa}}{n+1}\,, \tag{45}$$

$$\beta_1 = \phi, \quad \beta_2 = \frac{1}{n-2}(\phi^2 + \psi + (n-2)\mu),$$

$$\beta_3 = -\frac{\psi\phi}{n-2}, \quad \beta_4 = \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \frac{\psi}{n-2}\right)\phi, \tag{46}$$

$$R\cdot S = \frac{\widetilde{\kappa}}{n(n+1)}\, Q(g,S) - \phi\, Q(H,H^2), \tag{47}$$

$$\begin{aligned} C\cdot S &= \phi\, Q(H,S) + \frac{1}{n-2}(\phi^2 + \psi + (n-2)\mu)\, Q(g,S) \\ &\quad + (\frac{(n-1)\widetilde{\kappa}\phi}{n(n+1)} - \frac{1}{n-2}\,\psi\phi)\, Q(g,H), \end{aligned} \tag{48}$$

$$\begin{aligned} (n-2)\, R\cdot C &= (n-2)\, Q(S,R) + \phi\, g\wedge Q(H,H^2) \\ &\quad - \frac{(n-2)^2\widetilde{\kappa}}{n(n+1)}\, Q(g,R) - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, Q(S,G), \end{aligned} \tag{49}$$

$$\begin{aligned} (n-2)\, C\cdot R &= (n-3)\, Q(S,R) \\ &\quad + (\frac{\kappa}{n-1} + \psi - \frac{(n^2-3n+3)\widetilde{\kappa}}{n(n+1)})\, Q(g,R) \\ &\quad - \frac{(n-3)\widetilde{\kappa}}{n(n+1)}\, Q(S,G) + \phi\, H\wedge Q(g,H^2). \end{aligned} \tag{50}$$

Next, using (4) and (12), we find

$$H^2 = -S + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\, g, \tag{51}$$

$$H^4 = (\phi^2 + \psi)\, H^2 + \phi\psi\, H, \tag{52}$$

$$\begin{aligned} \phi\psi\, H &= S^2 - \left(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - \psi\right) S \\ &\quad + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\phi^2 + \psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) g. \end{aligned} \tag{53}$$

Further, (28) turns into

$$\begin{aligned} S^3 &= \left(\frac{3(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - 2\psi\right) S^2 + \left(\psi\left(\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \psi\right) - \left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)^2\right) S \\ &\quad + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\frac{(n-1)\widetilde{\kappa}}{n(n+1)}\left(\phi^2 + \psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right) - \psi\left(2\phi^2 + \psi - \frac{(n-1)\widetilde{\kappa}}{n(n+1)}\right)\right) g. \end{aligned} \tag{54}$$

We note that by the Gauss equation (11) and (53) we obtain on $\mathcal{U}_H$ the following relation

$$\begin{aligned} & 2(\phi\psi)^2\left(R - \frac{\widetilde{\kappa}}{n(n+1)}\, G\right) \\ = & \left(S^2 - (\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - \psi) S\right) \wedge \left(S^2 - (\frac{2(n-1)\widetilde{\kappa}}{n(n+1)} - \phi^2 - \psi) S\right). \end{aligned} \tag{55}$$

It is obvious that if the hypersurface $M$ in $N^{n+1}(c)$, $n \geq 4$, has at every point exactly three distinct principal curvatures then $M = \mathcal{U}_H$. In this case we also have $M = \mathcal{U}_S = \mathcal{U}_C$.

The above presented results lead immediately to the following proposition.

**Proposition 4.1.** *Let $M$ be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, having exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying at every point of $M$: $\lambda_1 = 0$, $\lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda \neq 0$. Then $M$ is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying (14) and (41)-(55).*

From the last proposition, (14) and (17) we immediately get the following.

**Proposition 4.2.** *Let $M$ be a hypersurface in an Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying at every point of $M$: $\lambda_1 = 0$, $\lambda_2 = -(n-2)\lambda$ and $\lambda_3 = \lambda_4 = \ldots = \lambda_n = \lambda \neq 0$. Then $M$ is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying (15), (43), (44) and*

$$
\begin{aligned}
S &= H^2, \quad \kappa = -tr(H^2) = -(n-2)(n-1)\lambda^2,\\
\phi\psi H &= S^2 + (\phi^2 + \psi)\, S,\\
S^3 &= -(\phi^2 + 2\psi)\, S^2 - \psi^2 S,\\
\phi &= -(n-3)\lambda, \quad \psi = (n-2)\lambda^2, \quad \mu = \frac{\kappa}{(n-2)(n-1)},\\
rank &\left(S - \frac{\kappa}{(n-2)(n-1)}\, g\right) = 2,
\end{aligned}
$$

$$
\begin{aligned}
R &= \frac{1}{2(\phi\psi)^2}\left(S^2 + (\phi^2+\psi)S\right) \wedge \left(S^2 + (\phi^2+\psi)S\right),\\
R\cdot S &= \phi\, Q(H,S) = \frac{n-1}{\kappa}\, Q(S,S^2),\\
C\cdot S &= \phi\, Q(H,S) + \frac{\phi^2}{n-2}\, Q(g,S) - \frac{\phi\psi}{n-2}\, Q(g,H)\\
&= \frac{n-1}{\kappa}\, Q(S - \frac{\kappa}{(n-2)(n-1)}\, g, S^2 - \frac{\kappa}{n-1}\, S),\\
(n-2)\, R\cdot C &= (n-2)\, Q(S,R) - \phi\, g \wedge Q(H,S),\\
(n-2)\, C\cdot R &= (n-3)\, Q(S,R) - \phi\, H \wedge Q(g,S).
\end{aligned}
$$

**Theorem 4.3.** *Let $M$ be a hypersurface in an Euclidean space $\mathbb{E}^5$ having exactly three distinct principal curvatures $\lambda_1$, $\lambda_2$ and $\lambda_3$ satisfying at every point of $M$: $\lambda_1 = 0$, $\lambda_2 = -2\lambda$ and $\lambda_3 = \lambda_4 = \lambda \neq 0$. Then $M$ is a minimal, 2-quasi-umbilical and 2-quasi-Einstein hypersurface satisfying: (15), (43), (44) and*

$$
\begin{aligned}
\lambda^2 &= -\frac{\kappa}{6}, \ \lambda H = \frac{3}{\kappa} S^2 - \frac{3}{2}\, S, \ H^2 = -S,\\
S^3 &= \frac{5\kappa}{6}\, S^2 - \frac{\kappa}{9}\, S\,, \ rank\,(S - \frac{\kappa}{6}\, g) = 2,\\
R &= -\frac{27}{\kappa^3}\left(S^2 - \frac{\kappa}{2}\, S\right) \wedge \left(S^2 - \frac{\kappa}{2}\, S\right),\\
R\cdot S &= \frac{3}{\kappa}\, Q(S,S^2),\\
C\cdot S &= \frac{3}{\kappa}\, Q(S - \frac{\kappa}{6}\, g, S^2 - \frac{\kappa}{3}\, S),\\
R\cdot C &= Q(S,R) + \frac{3}{2\kappa}\, g \wedge Q(S^2,S)\\
C\cdot R &= \frac{1}{2}\, Q(S,R) + \frac{3}{2\kappa}\, S^2 \wedge Q(g,S) - \frac{3}{8}\, Q(g, S\wedge S).
\end{aligned}
$$

**Example 4.1.** If $p = 1$ then the hypersurface $M$ defined in Example 3.2 (ii) has at every point three distinct principal curvatures $\lambda_1 = 0$, $\lambda_2 = \frac{1}{t}\rho_1$ and $\lambda_3 = \frac{1}{t}\rho_2$, of multiplicity 1, 1 and $n-2$, respectively. Further, we set $\lambda = \lambda_3 = \frac{1}{t}\rho_2 = \frac{1}{\sqrt{n-2}\,t}$. This by (38) yields $\lambda_2 = -(n-2)\lambda$. Thus we see that the cone over the Clifford torus $S^1(c_1) \times S^{n-2}(c_2)$, $c_1^{-1} = r_1 = \sqrt{\frac{1}{n-1}}$, $c_2^{-1} = r_2 = \sqrt{\frac{n-2}{n-1}}$, is a hypersurface in $\mathbb{E}^{n+1}$, $n \geq 4$, having exactly three distinct principal curvatures satisfying at every point (3).

**Example 4.2.** (i) Let $\mathcal{M}$ be a surface in $\mathbb{E}^{n+1}$, $n \geq 4$, given by the immersion $f : \mathcal{M} \to \mathbb{E}^{n+1}$ and $B\mathcal{M}$ be the tangent bundle of the unit normals to $\mathcal{M}$. The hypersurface $M$ defined by the map $\Phi_t : B\mathcal{M} \mapsto \mathbb{E}^{n+1}$, $\Phi_t(x, \xi) = F(x, t\xi) = f(x) + t\xi$, $t > 0$, is called the tube of radius $t$ over $\mathcal{M}$. If $\mu_1$ and $\mu_2$ are the principal curvatures of $\mathcal{M}$ then the principal curvatures of the tube $M$ are the following ([3]): $\lambda_1 = \frac{\mu_1}{1-t\mu_1}$, $\lambda_2 = \frac{\mu_2}{1-t\mu_2}$, $\lambda_3 = \lambda_4 = \ldots = \lambda_n = -\frac{1}{t}$. Clearly, (37) holds on $M$ ([13], Example 2).
(ii) In addition, we assume that the principal curvatures $\mu_1$ and $\mu_2 = \mu$ of $\mathcal{M}$ are constant, and $\mu_1 = 0$ and $\mu > 0$. Moreover, let $t = \frac{n-2}{(n-1)\mu}$. Now the principal curvatures of $M$ are the following: $\lambda_1 = 0$, $\lambda_2 = (n-1)\mu$, $\lambda_3 = -\frac{(n-1)\mu}{n-2}$ with multiplicity 1, 1 and $n-2$, respectively. Finally, if we set $\lambda = -\frac{(n-1)\mu}{n-2}$ then $\lambda_2 = -(n-2)\lambda$, and $\lambda_3 = \lambda$. Thus we see that (3) holds at every point of $M$.

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